Computing the **Straight Skeleton** of an **Orthogonal Monotone Polygon** in **Linear Time**

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Preliminaries

- $P$ is an orthogonal $x$-monotone polygon with $n$ vertices.
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- $P$ is an orthogonal $x$-monotone polygon with $n$ vertices.
- $S(P)$ denotes the straight skeleton of $P$.
- We split $P$ into its upper and lower monotone chain.
- Looking at a single chain $C$, let $S(C)$ denote its straight skeleton.
Algorithm Setup

The arcs of $S(C)$ have only three directions: $\left(\frac{1}{1}\right)$, $\left(-\frac{1}{1}\right)$, and $\left(\frac{0}{1}\right)$.
Algorithm Setup

A face $f(e_i)$ of $S(C)$ lies inside of the half-plane slab $\Pi_i$. 
Algorithm Setup

Also, $f(e_i)$ is monotone in respect its input edge as well as to a line perpendicular to it.
Algorithm Setup

Let us separate $f(e_i)$ into its left and right chain.
Algorithm Setup

We maintain the partial straight skeleton $S^*$ during our incremental construction. It contains the left chains of all edges already inserted.
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Algorithm Setup
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We maintain the partial straight skeleton $S^*$ during our incremental construction. It contains the left chains of all edges already inserted, as well as two stacks $R$ and $G$. 

$e_h$  
$e_i$  

$R$  
$G$
Constructing $S(C)$

We start our incremental construction by adding $e_1$. 
Constructing $S(C)$

The first arc $a$ of the left chain of $f(e_i)$ has $\binom{1}{1}$ or $\binom{-1}{1}$ direction.
Constructing $\mathcal{S}(C)$

The first arc $a$ of the left chain of $f(e_i)$ has $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ direction. It connects to the end of $f(e_{i-1})$'s left chain.
Constructing $S(C)$

Subsequent arcs between $e_i$ and the edge on top of $R$. 
Constructing $S(C)$

Subsequent arcs between $e_i$ and the edge on top of $R$. The last arc of a chain ends in a ray,
Constructing $S(C)$

Subsequent arcs between $e_i$ and the edge on top of $R$. The last arc of a chain ends in a ray, unfinished ghost arc,
Constructing $S(C)$

Subsequent arcs between $e_i$ and the edge on top of $R$. The last arc of a chain ends in a ray, unfinished ghost arc, or bounded vertical arc.
Arc $a$ has $\left(\frac{1}{1}\right)$ Direction

We follow with a case distinction for the next arc $a$ added in the left chain of $e_i$. Arc $a$ is a ray and we push $e_i$ onto $R$. 

\[
\begin{array}{c}
| e_i \\
| e_t \\
| R \\
| G \\
\end{array}
\]
Arc $a$ has $(\overline{-1})$ Direction

Arc $a$ is either a bounded arc or a ray.
Arc $a$ has $(-1)$ Direction

If the left chain of $e_{i-1}$ terminates in a bounded arc, and $a$ is the first arc on the left chain of $e_i$, it ends where the left chain of $e_{i-1}$ ends.
Arc $a$ has $(-1,1)$ Direction

Otherwise, we look at $e_t$ at the top of $R$. If $e_t$ does not terminate in a $(1,1)$ ray, $a$ is a $(-1,1)$ ray, $e_i$ is pushed onto $R$, and the chain is completed.
Arc $a$ has $(-1) \binom{1}{1}$ Direction

Otherwise, the left chain of $e_t$ terminates in a $\binom{1}{1}$ ray $r$. At $p$ arc $a$ intersects ray $r$. In $f(e_{i-1})$ we modify $r$ into a bounded arc $r'$ that ends at $p$, where $a$ ends as well.
Arc $a$ has $(-1)$ Direction

Finally we have to process the elements of $G$ below $r'$ and $a$. 
Arc $a$ has $(-1)^n$ Direction

Finally we have to process the elements of $G$ below $r'$ and $a$. 
Arc $a$ has $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ Direction

Arc $a$ is either a ghost arc or bounded vertical arc, starting at a point $p$. 
Arc $a$ has $\binom{0}{1}$ Direction

Arc $a$ is either a ghost arc or bounded vertical arc, starting at a point $p$. In case $a$ is a ghost arc we push $e_i$ onto $G$. 
Arc $a$ has $\binom{0}{1}$ Direction

Otherwise, $a$ is the line segment from $p$ that is contained in both $\Pi_t$ and $\Pi_i$. 
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Otherwise, $a$ is the line segment from $p$ that is contained in both $\Pi_t$ and $\Pi_i$. 

![Diagram showing the direction of arc $a$ and the line segments contained in $\Pi_t$ and $\Pi_i$.]
Finalizing $S(C)$

- We process the elements that remain on $G$. 
Finalizing $S(C)$

- We process the elements that remain on $G$.
- All arcs inserted intersect only rays or ghost arcs.
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**Theorem**

*Our incremental construction approach creates $S(C)$ in $\mathcal{O}(n)$ time.*
Skeleton Merging
Skeleton Merging

\[ f_u(1) \]

\[ f_l(1) \]

\[ a \]
Skeleton Merging

$fu(1)$

$f_l(1)$

$a$
Skeleton Merging

\[ f_u(1) \]

\[ f_i(j) \]
Skeleton Merging

\[ f_u(1) \]

\[ f_i(j) \]
**Skeleton Merging**

\[ f_u(i) \]

\[ f_l(j) \]
Skeleton Merging

\[ f_u(i) \]

\[ f_i(j) \]
Skeleton Merging

\[ f_u(i) \]

\[ f_l(j) \]

\[ a \]
Skeleton Merging
Skeleton Merging
Skeleton Merging
Summary

- Incremental construction of $S(C)$ in linear time.
- Merge of both straight skeletons in linear time.

Questions?